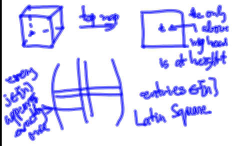


top map

the only 1 above my head is at height t

every $j \in [n]$ appears exactly once

entries $\in [n]$

Latin Square

aug 7-21:02

$$(n!)^{1/n} = (1+o(1))\frac{n}{e}$$

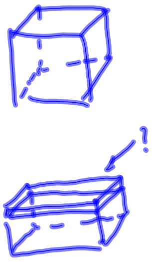
$$L_n^{1/n^2} = (1+o(1))\frac{n}{e^2}$$

Conj: $\forall d \geq 1$ $\left(S_n^d\right)^{1/n^d} \overset{??}{=} (1+o(1))\frac{n}{e^d}$

Thm (w/ Zur Luria) $\leq$

$\geq$ const easy

aug 8-08:47



Every slice/layer is a perm matrix

possibilities for the next layer =

= per B "no "1" below me".

"I see a 1 below me" $\rightarrow 0$

aug 8-09:40

There is no high-dim analog of v.d.W conj.

From the marriage thm $\Rightarrow$ if a 0/1 matrix has exactly k 1's in every row and every column $\rightarrow$ per > 0.

$\Leftrightarrow$ A regular bipartite graph has a perfect matching.

aug 8-09:45

von-Neumann Birkhoff Thm:

Let $\Omega_n = \{$all $n \times n$ doubly-stoch. matrices$\}$

Ω_n is a convex polytope.

v.N-B. Thm: Permutation matrices are the vertices of Ω_n.

aug 8-09:52

Triply stochastic array:

$a_{ijk} \geq 0$

$\forall i,j$ $\sum_k a_{ijk} = 1$

$\forall i,k$ $\sum_j a_{ijk} = 1$

$\forall j,k$ $\sum_i a_{ijk} = 1$

$\Omega_n^{(3)}$

2-dim perm is a vertex of $\Omega_n^{(3)}$ but there are more vertices.

aug 8-09:55

Our approach to the upper bd

$$(S_n^d)^{1/n^d} \le (1+o(1))\frac{n}{e^d}$$

L. Schrijver | Brégman
J. Radhakrishnan

aug 8-09:58

Pf of Brégman's Thm

Entropy method

X is a discrete random var with probs $p_1,\dots,p_N>0$

$$H(X):=-\sum p_i \log p_i$$

aug 8-10:04

Several simple properties of the entropy function: Chain rule

$$H(X_1,\dots,X_n)=H(X_1)+H(X_2|X_1)+H(X_3|X_1,X_2)+\dots$$

$H(X|Y)$

$$\left(\begin{array}{c} \vdots \\ p_{ij} \\ \vdots \end{array} \right) \ne \Pr\begin{pmatrix} X=i \\ Y=j \end{pmatrix}$$

aug 8-10:07

If $|\mathrm{dom}\, X| = N \Rightarrow H(X) \le \log N$ w/ equality iff X is uniformly distributed.

aug 8-10:10

A way of estimating $|S|$:

Consider a r.v. X that is uniformly distributed over S

$$H(X) = \log |S|.$$

aug 8-10:12

Given a 0/1 matrix A with row sums $r_1,\dots,r_n$, let X be a uniformly chosen generalized diagonal (of 1's) in A.

$$H(X) = \log \operatorname{per} A$$

$S :=$ set of gen. diags. of 1's in A.

aug 8-10:14

(X = unif. gen. diag. of 1's in A).

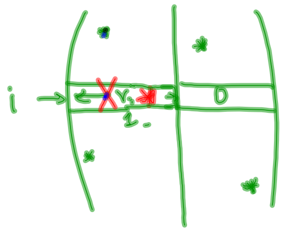
X_i = the position of the 1 in the i-th row of A

$A = \begin{pmatrix} & * & & \\ * & & & * \\ X_1 \to & & * & \\ \end{pmatrix}$

$H(X) = H(X_1) + H(X_2 | X_1) + \dots$

aug 8-10:17

?? $\mathbb{E}_\sigma H(X_i | X_1, \dots, X_{i-1}) \le \mathbb{E}_\sigma \log N_i$



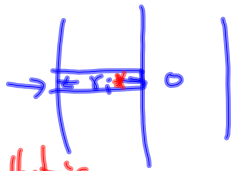
When we get to the i-th row there are N_i one's that are still available to us

aug 8-10:20

N_i = # non-shaded 1's in the i-th row

Instead of working in the order $1, \dots, n$ we select a random ordering σ of the rows.

aug 8-10:24



You have invited r_i guests to a party, one of them is your guest of honor

What is $E_\sigma \log N_i$?

Guests are arriving in random order (σ_i). There are N_i guests that come after your g.o.h. (including the g.o.h.).

aug 8-10:28

$$\mathbb{E} \log N_i = \frac{1}{r_i}(\log 1 + \log 2 + \dots + \log r_i) = \frac{\log r_i!}{r_i} = \log (r_i!)^{1/r_i}$$

aug 8-10:31

Geom. av. of the numbers $1, \dots, k$ is $\sim \frac{k}{e}$

$$(S_d^n)^{1/nd} \overset{?}{=} (1+o(1)) \frac{n}{e^d}$$

Know $\le$

aug 9-08:44

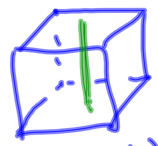
Define for A a $[n]^{d+1}$ array of 0/1

$\mathrm{per}_d A = \#$ d-permutations that are included in A

$\mathrm{per}_1 \equiv \mathrm{per}$

aug 9-11:10

We will consider only one type of lines in A and let r_i be the # of 1's in the i-th row

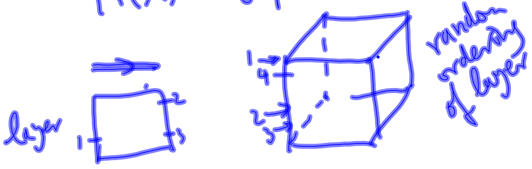


Thm: $\mathrm{per}_d(A) \le \prod_i \exp(f_d(r_i))$

aug 9-11:12

Again let X be a unif. dist. d-perm included in A

$H(X) = \lg \mathrm{per}_d A$



aug 9-11:16

$f_0(r) = \lg r$

$f_d(r) = \frac{1}{r} \sum_{k=1}^{r} f_{d-1}(k)$

W/ Zur Luria: get upper bds on # Steiner Triple systems
1-Factorizations
(Presumably these bds are tight).

aug 9-11:20

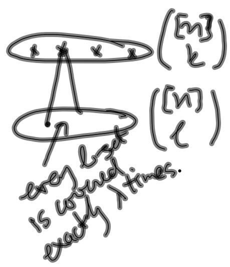
$\underline{d=2}$ Symmetric matrix

$\forall_{ij} \quad a_{ij} = a_{ji}$

in 3D $\quad a_{ijk} = a_{ikj} \dots = a_{kji}$
6 possibilities.

$\sim$ A collection of triples that cover every pair exactly once
"STS"

aug 9-11:23

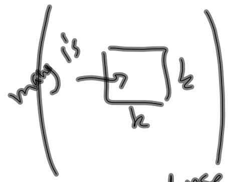


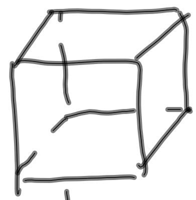
$k > \ell$ fixed
$n \to \infty$

aug 9-11:27

Random generation of comb. objects
Create a rapidly mixing Markov chain on such objects
Jacobson & Matthews ⇒ MC on LS's.
connected, rapid mixing?
"What does a random LS look like?"

aug 9-11:29





dense

Find $k \times k \times k$ subarrays $\geq k$

$\frac{k^3}{n}$

$k \approx \sqrt{n}$

aug 9-11:33

Q: Are there $n \times n \times n$ ~~of arrays~~
(= i.e. 2-per's)
s.t. in every $\sqrt{n} \times \sqrt{n} \times \sqrt{n}$ subarray there are only $O(\sqrt{n})$ 1's?
Even more: Is this the typical situation.

aug 9-11:36

Simplicial complexes (mostly random...).

X a simplicial complex with vertex set V is just a family of subsets of V with:

$A \in X,\ B \subseteq A \Rightarrow B \in X$

members of X are called faces/simplices

aug 9-11:39

$A \in X$, $\dim A := |A| - 1$

$\dim X := \max_{A \in X} \dim A$

Graph = 1-dim simplicial complex

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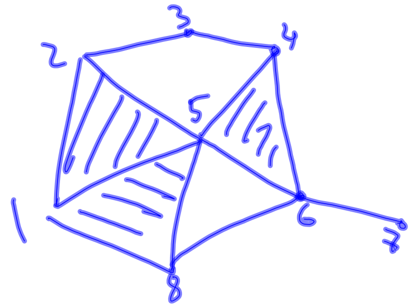

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Extremal graph theory

⇒ Random Graphs

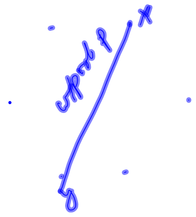
Turán's thm: If a graph has more than $\frac{r-2}{r-1}\binom{n}{2}$ edges ⇒ it contains a K_{r+1}

the result is tight.

aug 9-11:45

$G(n,p)$ Erdős–Rényi model

often $p=p(n)$



n vertices

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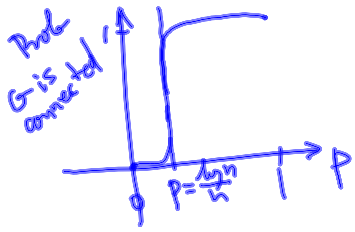
Threshold phenomena

P = "being connected"

E. Friedgut



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why is $p_{crit,\ connectivity} \geq \frac{\log n}{n}$?

Coupon collector idea from prob. theory: if $p<(1-\varepsilon)\frac{\log n}{n}$ $\exists$ w almost certainly isolated vertices

For $p>(1+\varepsilon)\frac{\log n}{n}$ almost surely G is connected.

aug 9-11:58

How to interpret graph connectivity using linear algebra?

$$V\ \begin{array}{c} \\ x \\ y \end{array}\!\!\left(\begin{array}{c} xy \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}\right)$$

E

incidence matrix

$\mathbb{F}_2$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

aug 9-12:02

$$\begin{array}{cc} & E \\ V & \begin{pmatrix} \quad A \quad \end{pmatrix} \end{array}$$

$\mathbb{F}_2$

left kernel $=$ $\{x \mid xA=0\}$.

Always $\vec{1}$ left kernel

$=$ span $\langle \chi_{\text{connected cpts of } G} \rangle$

$G =$ 1s O O O

aug 9-12:05

G is connected $\Leftrightarrow$ left ker(A) is trivial (it consists only of $\vec{1}$).

$p\binom{n}{2}$ $\{i,j\}$

n

Initially left ker $= \vec{1}$

for small enough p we expect to have a non-trivial left kernel

crit $p = \frac{\log n}{n}$

aug 9-12:08

$p\binom{n}{3}$ ijk

$\binom{n}{2}$ ij, ik, jk

0

Q: What is the critical p for which this resulting random matrix has a non-trivial left kernel? $\mathbb{F}_2$

$n\left(\text{incidence}\right)\binom{n}{2}\left(\text{incidence}\right) =$

$\binom{n}{2}$ $\binom{n}{3}$

aug 9-12:11

$\boxed{\partial\partial = 0}$ ijk, ij, ik, i

row space $n\left(\ \right)_{\binom{n}{2}}$ $\subseteq$ left ker $\binom{n}{2}\left(\ \right)^{p\binom{n}{3}}$ $= 0$

aug 9-12:16

$AB = 0$ left ker B $\supseteq$ row space (A)

is this an equality

Q: What is the crit p s.t. the only vectors in the left ker $\binom{n}{2}\left(\ \right)_{p\binom{n}{3}}$ come from row space $n\left(\ \right)^{\binom{n}{2}}$

aug 9-12:18

Answer (L + Meshulam) crit $p = \frac{2\log n}{n}$

$\geq$ easy

$p\binom{n}{3}$ $p\binom{n}{2}$

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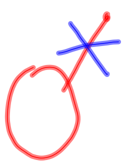
Let G be an n-vertex graph with $n-1$ edges, then TFAE

1) G has no cycles

2) G is connected

3) G is collapsible



Elem. collapse

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$1 \iff 2$ $\quad V\overset{E}{\left(\qquad\right)}_{n}$ $\quad n-1$

G connected $\iff$ left ker $= \langle 1 \rangle$

$\iff$ rank $= n-1$

$\iff$ right kernel empty

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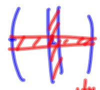

collapsing

empty matrix

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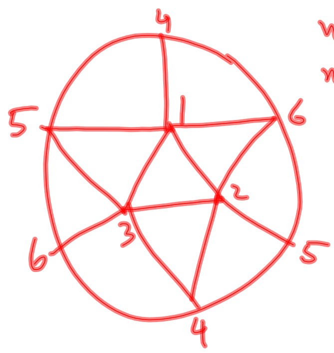

not collapsible

non zero right kernel over $\mathbb{F}_2$

No right kernel over $\mathbb{Q}$

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